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Spectral Properties of Burgers and KPZ Turbulence

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This paper presents the higher-order spectral densities of non-Gaussian random fields arising as scaling limits in the Burgers and KPZ turbulence problems with strongly dependent non-Gaussian initial conditions.

KEY WORDS: Burgers' equation, the KPZ equation, scaling laws, higher-order spectral densities, long-range dependence.

1. INTRODUCTION

The Burgers equation provides an important model of hydrodynamical turbulence. It has been used to describe a variety of nonlinear phenomena in wave propagation, acoustics and plasma physics (see, for example,[16,17,31,55,59]). The books[23,26,34,60] contain an extensive bibliography of the subject and an exposition of some key results of the theory of Burgers turbulence.

The Burgers equation with random initial conditions has been extensively studied.[13–15,18,19,24,29,35,37,46,51,53,56] Gaussian and non-Gaussian scenarios for parabolically rescaled solutions of the Burgers equation under weakly dependent or strongly dependent random initial conditions have been studied in.[1,15,18,24,35,40–43] These scenarios are in some sense subordinated to the Gaussian white noise measure. Further related problems have also been investigated; these include asymptotic distributions of averages of solutions of the Burgers equation with random data,[11,29,51] statistics of shocks and related topics,[56]

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hyperbolic asymptotics,[46] large deviation principle and statistics of shock waves.[13,14,53]

A related equation which plays an important role in describing the evolution of the profile of a growing interface is the Kardar-Parisi-Zhang (KPZ) equation (see[10,31,60]). The basic form of the KPZ equation for interface elevation is obtained from the heat equation via a log transform (see Sec. 3), while the gradient of this elevation follows the Burgers equation[60] p. 10. A construction of the KPZ equation via an approach involving chemical potential was detailed in.[32] The KPZ equation with long-range interactions was investigated in,[30,33,48] while the KPZ equation under the additional possibility that surface transport may be effected via a hopping mechanism of a Lévy flight was studied in.[44]

Gurbatov *et al.*[27] studied the decay of the random solutions of the unforced Burgers equation in one dimension in the limit of vanishing viscosity. In particular, they investigated the case when the initial viscosity is homogeneous and Gaussian with a spectral density proportional to (in our notation) $|\lambda|^{\kappa}$ at small $|\lambda|$, where $-1 < \kappa < 2$. At large times, they obtained three scaling regions of singularity of the solutions. On the other hand, Gurbatov[25] studied the distributional non-Gaussian properties of the unforced multidimensional Burgers and KPZ equations in the limit of vanishing viscosity. It should be noted that vanishing viscosity corresponds to hyperbolically rescaled solutions of the equations.

In this paper, we will be concerned with parabolically rescaled solutions of Burgers and KPZ equations. These parabolically rescaled solutions are in fact approximations to the hyperbolically rescaled solutions. We present the second- and higher-order spectral densities of homogeneous (in space) random fields arising as rescaled solutions of the Burgers and KPZ equations with singular non-Gaussian initial conditions. This work is a continuation of those by Leonenko and Woyczynski,[37–39] in which the second-order spectral densities were studied for the Burgers turbulence problem with non-Gaussian singular data, and Anh, Leonenko and Sakhno,[6] in which second- and higher-order spectral densities were given for fractional random fields arising as rescaled solutions of the heat and fractional heat equations with singular random data (for further details on these equations, see[2–4]).

In a sense, non-Gaussian scenarios are more realistic models of zero viscosity than Gaussian scenarios. Furthermore, to provide a full description of singularity, we have to consider higher-order spectral densities and their singular properties (see Sec. 2 for Burgers turbulence and Sec. 3 for KPZ turbulence). But even for the second order, our results for the spectral density in one dimension can be compared with the results of.[27] Indeed the singular property of the energy spectrum of the initial condition (2.9) is transformed by the Burgers equation into the singular property (2.13), which for $n = 1$, $l = k = 1$ and up to a constant reads $|\lambda|^{\kappa} e^{-2\mu t \lambda^2}$, $\kappa = 2\alpha + 1$, $0 < \alpha < 1/2$. This result is exactly the same as formula (122) of.[27] However, we can see from (2.13) that these singular properties depend on the dimension n, and the results change dramatically starting from dimension $n \geq 3$. In our opinion, both non-Gaussian scenarios in parabolically rescaled Burgers and KPZ equations and singular properties of higher-order spectral densities provide a description of Burgers and KPZ turbulence complementary to that of[25,27] via vanishing viscosity together with a power-law investigation of the solutions.

The closed-form expressions of higher-order spectral densities in turn will play an essential role in the statistical estimation of these random fields. In fact, in the presence of possible long-range dependence, non-Gaussianity and non-linearity inherent in the formulated models, particularly in a situation where useful information is contained in higher orders rather than the second order, an estimation theory using information in higher-order spectral densities is more viable. Some components of such a theory are provided in[7–9] based on the minimum contrast principle.

2. NON-GAUSSIAN SCENARIOS IN BURGERS TURBULENCE AND THEIR SPECTRA

Consider the n-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + (u, \nabla)u = \mu \Delta u, \quad \mu > 0, \tag{2.1}$$

subject to the random initial conditions in potential form:

$$u(0, x) = \nabla \eta(x),\ x \in \mathbb{R}^n, \tag{2.2}$$

where Δ denotes the n-dimensional Laplacian and ∇ the gradient operator in $\mathbb{R}^n$. Equation (2.1) describes the time evolution of the velocity field

$$u(t, x) = (u_1(t, x), \ldots, u_n(t, x)),\ (t,\ x) \in (0, \infty) \times \mathbb{R}^n, n \geq 1.$$

We will assume that the initial velocity potential $\eta(x)$ is a scalar random field of the form described in Condition A below.

Equation (2.1) is a parabolic equation with quadratic, inertial nonlinearity, which can be viewed as a simplified version of the Navier-Stokes equation with the pressure term ∇p omitted, and with the viscosity coefficient μ corresponding to the inverse of the Reynolds number (see[51], p. 152). With random initial data, the problem (2.1)–(2.2) is also known as the Burgers turbulence problem.

Via the Cole-Hopf transformation

$$u(t, x) = -2\mu \nabla \log h(t, x), \tag{2.3}$$

the Burgers problem (2.1)–(2.2) is reduced to the parabolic-type equation

$$\frac{\partial h}{\partial t} = \mu \Delta h, \; t > 0, \; x \in \mathbb{R}^n \tag{2.4}$$

subject to the initial condition

$$h(0, x) = h_0(x) = \exp\left\{-\frac{\eta(x)}{2\mu}\right\} \tag{2.5}$$

(see e.g.[26,59]).

The fundamental solution to (2.4) is of the form

$$h(t, x) = \frac{1}{(4\pi\mu t)^{n/2}} \exp\left\{-\frac{\|x\|^2}{4\mu t}\right\}, \; t > 0, \; x \in \mathbb{R}^n. \tag{2.6}$$

Thus, the field

$$u(t, x) = \frac{\int_{\mathbb{R}^n} \frac{x-y}{t} h(t, x-y) e^{-\frac{\eta(y)}{2\mu}} dy}{\int_{\mathbb{R}^n} h(t, x-y) e^{-\frac{\eta(y)}{2\mu}} dy} \tag{2.7}$$

solves the initial-value problem (2.1)–(2.3).

We now introduce the following condition concerning the initial velocity potential.

A. The initial velocity potential $\eta(x)$ is a random field of the form

$$\eta(x) = \xi^2(x) - 1, \; x \in \mathbb{R}^n,$$

where the random field $\xi(x)$ is a real measurable homogeneous and isotropic Gaussian field with $E\xi(x) = 0$, $E\xi^2(x) = 1$ and covariance function of the form

$$B(x) = \|x\|^{-\alpha} L(\|x\|), \; 0 < \alpha < n, \; as \; x \to \infty, \tag{2.8}$$

where the function $L(t)$, $t > 0$, is slowly varying at infinity and is bounded on each bounded interval. Furthermore, the spectral density $f(\lambda)$, $\lambda \in \mathbb{R}^n$, of the field $\xi(x)$ exists, is decreasing for $\|\lambda\| \geq \lambda_0 > 0$ and continuous for all $\lambda \neq 0$.

Noting that the random field $\xi(x)$ of Condition A can be represented as

$$\xi(x) = \int_{\mathbb{R}^n} e^{i(\lambda, x)} \sqrt{f(\lambda)} W(d\lambda),$$

where $W(\cdot)$ is a Gaussian white noise, and from the Tauberian theorem for Hankel type transform (see, for instance,[34] Theorem 1.1.4), we obtain that the spectral density $f(\lambda)$ satisfies

$$f(\|\lambda\|) \sim \|\lambda\|^{\alpha - n} L\left(\frac{1}{\|\lambda\|}\right) c(n, \alpha), \; 0 < \alpha < n, \; \|\lambda\| \to 0, \tag{2.9}$$

where $c(n, \alpha)$ is the Tauberian constant

$$c(n, \alpha) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^{\alpha}\pi^{n/2}\Gamma(\alpha/2)}. \tag{2.10}$$

The result (2.9) means that the initial condition under consideration displays a singular property; in fact, the random field $\xi(x)$ will then have long-range dependence.

We will study the spectral properties of the limit distributions of the rescaled solutions, namely, with parabolic scaling, of the Burgers equation (2.1) with initial data (2.2) satisfying Condition A. These parabolic scaling limits of the solution can be described in terms of their multiple stochastic integral representation as stated in the following theorem (see[34,35,41]).

Theorem 1. *Let $u(t, x)$, $(t, x) \in (0, \infty) \times \mathbb{R}^n$, be a solution of the initial value problem (2.1)–(2.2) with the random initial condition $\eta(x) = \xi^2(x) - 1$ satisfying Condition A and $\alpha \in (0, n/2)$. Then the finite-dimensional distributions of the random fields*

$$Z_\varepsilon(t, x) = \frac{\varepsilon^{-(1+\alpha)/2}}{L(1/\sqrt{\varepsilon})} u\left(t/\varepsilon, x/\sqrt{\varepsilon}\right), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, 0 < \alpha < n/2,$$

converge weakly, as $\varepsilon \longrightarrow 0$, to the finite-dimensional distributions of the vector field $Z_1(t, x)$, $(t, x) \in (0, \infty) \times \mathbb{R}^n$, homogeneous in x, with the following multiple stochastic integral representation:

$$Z_1(t, x) = C(\mu)c(n, \alpha) \int'_{\mathbb{R}^{2n}} \frac{e^{i(x, \lambda_1+\lambda_2)-\mu t\|\lambda_1+\lambda_2\|^2}(\lambda_1 + \lambda_2)}{(\|\lambda_1\|\,\|\lambda_2\|)^{(n-\alpha)/2}} W(d\lambda_1)W(d\lambda_2), \tag{2.11}$$

$(t, x) \in (0, \infty) \times \mathbb{R}^n, 0 < \alpha < n/2$, *where the constant $c(n, \alpha)$ is given by (2.10),*

$$C(\mu) = \frac{\mu^2 i}{1 + \mu}, \tag{2.12}$$

and the double stochastic integral $\int' \ldots$ is evaluated with respect to the Gaussian complex white noise measure $W(\cdot)$ in $\mathbb{R}^n$ with the diagonal hyperplanes $\lambda_1 = \pm\lambda_2$ being excluded from the domain of the integration.

We now describe the second-order and higher-order spectral densities of the non-Gaussian vector random field $Z_1(t, x)$ representing the limit of the parabolically rescaled solution of the problem (2.1)–(2.2). Note that different non-Gaussian scenarios are also given in.[24,40,42,43]

Let us recall firstly the definition of the cumulant spectra of order $k \geq 2$ of a vector-valued strictly stationary mean-zero continuous-parameter random field

$Z(x) = \{Z_1(x), \ldots, Z_p(x)\}$, $x \in \mathbb{R}^n$. We will suppose that the moments of all orders of $Z_j(x)$, $j = 1, 2, \ldots, p$ exist and define

$$c_{l_1 \ldots l_k}(x_1, \ldots, x_k) = \frac{1}{i^k} \frac{\partial^k}{\partial u_1 \ldots \partial u_k} \log E \exp \left\{ i \sum_{j=1}^{k} u_j Z_{l_j}(x_j) \right\} \Bigg|_{u_1 = \ldots = u_k = 0}$$
$$= cum \left\{ Z_{l_1}(x_1), \ldots, Z_{l_k}(x_k) \right\},$$

$x_1, \ldots, x_k \in \mathbb{R}^n$, $1 \leq l_i \leq p, i = 1, \ldots, k, k \geq 2$. In view of the strict stationarity of the field $Z(x)$, this cumulant function satisfies

$$c_{l_1 \ldots l_k}(x_1, \ldots, x_k) = c_{l_1 \ldots l_k}(x_1 - x_k, \ldots, x_{k-1} - x_k, 0).$$

The cumulant spectra of order k for the field $Z(x)$ are defined as complex-valued integrable functions

$$f_{l_1 \ldots l_k}(\lambda_1, \ldots, \lambda_{k-1}) \in L_1(\mathbb{R}^{(k-1)n})$$

such that

$$c_{l_1 \ldots l_k}(x_1 - x_k, \ldots, x_{k-1} - x_k, 0) = \int_{\mathbb{R}^{(k-1)n}} \exp \left\{ i \sum_{j=1}^{k-1} (\lambda_j, x_j - x_k) \right\}$$
$$\times f_{l_1 \ldots l_k}(\lambda_1, \ldots, \lambda_{k-1}) d\lambda_1 \ldots d\lambda_{k-1},$$

provided that such functions exist.

In the following, we will need to consider the symmetrized version of a function of $k - 1$ variables $f(\lambda_1, \ldots, \lambda_{k-1})$, where symmetrization is taken over k variables $\lambda_1, \ldots, \lambda_k$ such that $\sum_{j=1}^{k} \lambda_j = 0$. This symmetrized version is defined as

$$\underset{\{\lambda_1, \ldots, \lambda_k : \lambda_1 + \ldots + \lambda_{k-1} + \lambda_k = 0\}}{\text{sym}} f(\lambda_1, \ldots, \lambda_{k-1}) = \frac{1}{k!} \sum_{\pi \in \mathcal{P}_k} f(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(k-1)}),$$

where $\mathcal{P}_k$ is the set of all $k!$ permutations $\pi = (\pi(1), \ldots, \pi(k))$ of the set $\{1, \ldots, k\}$ and the variables $\lambda_1, \ldots, \lambda_k$ satisfy the restriction $\sum_{j=1}^{k} \lambda_j = 0$.

The second-order and higher-order spectra of the non-Gaussian random field $Z_1(t, x)$ are presented in the next theorem.

Theorem 2. *The random field* $Z_1(t, x) = (Z_1^{(1)}(t, x), \ldots, Z_1^{(n)}(t, x))$, $(t, x) \in (0, \infty) \times \mathbb{R}^n$, *defined by the representation (2.11) with fixed* $t > 0$ *is strictly stationary in* x. *Its moments of all orders exist and the corresponding cumulant spectra can be expressed as follows.*

(a) The spectral densities of second order are given by

$$f_{lk}(\lambda) = \frac{\mu^4 c^2(n,\alpha)}{(1+\mu)^2} k(\alpha) e^{-2\mu t\|\lambda\|^2} \|\lambda\|^{2\alpha-n} \lambda^{(l)} \lambda^{(k)},$$

$$l, k = 1, \dots, n, \lambda = (\lambda^{(1)}, \dots, \lambda^{(n)}) \in \mathbb{R}^n. \tag{2.13}$$

(b) The spectral densities of order $k \geq 3$ are given by

$$f_{l_1 \dots l_k}(\lambda_1, \dots, \lambda_{k-1}) = \left(\frac{\mu^2 i}{(1+\mu)} \right)^k c^k(n,\alpha)\, 2^k (k-1)$$

$$\times \underset{\{\lambda_1,\dots,\lambda_k : \Sigma_{i=1}^k \lambda_i = 0\}}{\mathrm{sym}} \left\{ h_{l_1 \dots l_k}(\lambda_1, \dots, \lambda_{k-1}) \right\},$$

$$1 \leq l_i \leq n, i = 1, \dots, k,\ \lambda_i = \left(\lambda_i^{(1)}, \dots, \lambda_i^{(n)}\right) \in \mathbb{R}^n, \tag{2.14}$$

where

$$h_{l_1 \dots l_k}(\lambda_1, \dots, \lambda_{k-1}) = \exp\left\{ -\mu t \left(\sum_{i=1}^{k-1} \|\lambda_i\|^2 + \|\sum_{i=1}^{k-1} \lambda_i\|^2 \right) \right\}$$

$$\times \lambda_1^{(l_1)} \lambda_2^{(l_2)} \dots \lambda_{k-1}^{(l_{k-1})} \left(\sum_{i=1}^{k-1} \lambda_i^{(l_k)} \right) g_k(\lambda_1, \dots, \lambda_{k-1}).$$

Here we have denoted

$$k(\alpha) = \pi^{n/2} \left\{ \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \right\}^2 \frac{\Gamma\left(\frac{n}{2} - \alpha\right)}{\Gamma(\alpha)}, \tag{2.15}$$

$$g_k(\lambda_1, \dots, \lambda_{k-1}) = \int_{\mathbb{R}^n} \frac{d\lambda}{\left(\|\lambda\| \|\lambda + \lambda_1\| \dots \left\|\lambda + \Sigma_{i=1}^{k-1} \lambda_i\right\|\right)^{n-\alpha}}. \tag{2.16}$$

From Theorem 2 one can see that the limiting random field $Z_1(t, x)$ of the rescaled solutions to (2.1)–(2.2) has singular properties. Let us now look at these properties in more details.

From the formula (2.13), the singular properties of the second-order spectral densities can be deduced, namely, the matrix of the second-order spectral densities satisfies

for $n = 1, 2$,

$$\lim_{\|\lambda\| \to 0} tr \{f_{lk}(\lambda)\}_{l,k=1,\dots,n} = 0; \tag{2.17}$$

for $n \geq 3$,

$$\lim_{\|\lambda\| \to 0} \left(\frac{2\mu^4 c^2(n,\alpha)}{(1+\mu)^2} k(\alpha) \right)^{-1} tr \{f_{lk}(\lambda)\}_{l,k=1,\dots,n} = \begin{cases} 0, & \frac{n}{2} - 1 < \alpha < \frac{n}{2}, \\ 1, & \alpha = \frac{n}{2} - 1, \\ \infty, & 0 < \alpha < \frac{n}{2} - 1, \end{cases}$$

and we can also conclude that, for each element of the matrix of the second-order spectral densities $\{f_{lk}(\lambda)\}_{l,k=1,\dots,n}$, the same behavior at the origin holds.

Remark 1. The assertion that all the moments of the random field $Z_1(t,x) = (Z_1^{(1)}(t,x), \dots, Z_1^{(n)}(t,x))$, $(t,x) \in (0,\infty) \times \mathbb{R}^n$, are finite follows from the general theory of multiple stochastic integrals. We also note the following inequalities due to McKean[45] and Nualart, Ustunel and Zakai[49] respectively:

$$E\left|Z_1^{(i)}(t,x)\right|^{2r} \leq \left(\frac{2r}{r2^r}\right)^2 \left[E\left(Z_1^{(i)}(t,x)\right)^2\right]^r,$$

and

$$E\left|Z_1^{(i)}(t,x)\right|^r \leq (r-1)^r \left[E\left(Z_1^{(i)}(t,x)\right)^2\right]^{r/2}$$

(and, evidently, analogous inequalities can be obtained for the mixed moments).

Properties of the spectral densities of order $k \geq 3$ of the random field $Z_1(t,x)$ can be deduced from their representation (2.14) where the singular integrals (2.16) are involved. Consider, for example, the case $k = 3$. Using the Riesz composition formula (see Appendix A), we have for the function $g_3(\lambda_1, \lambda_2)$:

$$\begin{aligned} g_3(\lambda_1,\lambda_2) &= \int_{\mathbb{R}^n} (\|\lambda\| \, \|\lambda+\lambda_1\| \, \|\lambda+\lambda_1+\lambda_2\|)^{\alpha-n} \, d\lambda \\ &\leq \left(\int_{\mathbb{R}^n} (\|\lambda\| \, \|\lambda+\lambda_1\|)^{3(\alpha-n)/2} \, d\lambda\right)^{1/3} \\ &\leq \left(\int_{\mathbb{R}^n} (\|\lambda\| \, \|\lambda+\lambda_1+\lambda_2\|)^{3(\alpha-n)/2} \, d\lambda\right)^{1/3} \\ &\leq \left(\int_{\mathbb{R}^n} (\|\lambda+\lambda_1\| \, \|\lambda+\lambda_1+\lambda_2\|)^{3(\alpha-n)/2} \, d\lambda\right)^{1/3} \\ &\leq k\left(\frac{3}{2}\alpha - \frac{n}{2}\right) (\|\lambda_1\| \, \|\lambda_2\| \, \|\lambda_1+\lambda_2\|)^{(3\alpha-2n)/3} \end{aligned}$$

for $\frac{n}{3} < \alpha < \frac{n}{2}$. The last inequality enables us to conclude, in particular, that

$$when\ \ \lambda_1 \to 0,\ g_3(\lambda_1,\lambda_2) = O(\|\lambda_1\|^{\alpha-2n/3}),$$

$$when\ \ \lambda_2 \to 0,\ g_3(\lambda_1,\lambda_2) = O(\|\lambda_2\|^{\alpha-2n/3}),$$

$$andwhen\ \ \lambda \to 0,\ g_3(\lambda,\lambda) = O(\|\lambda\|^{3\alpha-2n}).$$

Note that the functions $g_k(\lambda_1, \dots, \lambda_{k-1})$ represented by the formula (2.16) as singular integrals will also appear in the next section in the expressions for higher-order spectral densities for approximations of rescaled solutions of KPZ equations with singular non-Gaussian initial conditions. The functions $g_k(\lambda_1, \ldots, \lambda_{k-1})$ and similar functions had also been used in the description of higher order spectral densities for approximations of the rescaled solutions of the heat and fractional heat equations with singular random data in.[6] We should note some inaccuracies which occurred there when evaluating the behavior of these functions. In fact, Remarks 4, 5 and the second part of Remark 6 concerning the case $p \geq 3$ should be disregarded from the exposition. Instead, the description of the behavior of the functions $g_{m,3}$ in Remarks 4 and 6 for the range $\frac{2n}{3m} < \kappa < \frac{n}{m}$ should be done in the same manner as for the functions g_3 above. We also note a misprint in Remark 6: the factor $e^{-2\mu t\|\lambda\|^2}$ is missing in the expression for the spectral density $S_{2,2}(\lambda)$.

Let us now consider the class of non-Gaussian limiting distributions of the solution to the initial value problem (2.1)–(2.2) in the case where the initial velocity potential is a χ^2-field of degree p with long-range dependence described in the following condition.

B. The initial velocity potential $\eta(x)$ is a random field of the form

$$\eta(x) = \eta_p(x) = \frac{1}{2}\sum_{i=1}^{p}\left(\xi_i^2(x) - 1\right), \quad x \in \mathbb{R}^n,$$

where $\xi(x) = (\xi_1(x), \ldots, \xi_p(x))'$, $x \in \mathbb{R}^n$, is a real measurable homogeneous isotropic almost surely differentiable vector Gaussian field with $E\xi(x) = 0$ and covariance matrix

$$E\xi(0)\xi(x)' = (B_{ij}(\|x\|))_{1\leq i,j\leq p},$$

with

$$B_{ii}(\|x\|) = a(\|x\|), \quad i = 1, \ldots, p,$$

$$B_{ij}(\|x\|) = b(\|x\|), \quad i \neq j,\ i, j = 1, \ldots, p,$$

and

$a(0) = 1$, $b(0) = \rho_0 \in [0, 1)$;

$a(\|x\|) = \|x\|^{-\alpha}L(\|x\|)$, $b(\|x\|) = \rho_\infty\|x\|^{-\alpha}L(\|x\|)$ as $\|x\| \to \infty, \rho_\infty \in [0, 1), \alpha > 0$.

Here, the function $L(t)$, $t > 0$, is slowly varying at infinity and is bounded on each bounded interval.

Remark 2. If the constant $\rho_\infty = 0$, then $\xi_1(x), \ldots, \xi_p(x)$ are independent copies of the Gaussian random field $\xi(x)$ satisfying Condition A.

Note that the random field $\eta_p(x)$ of Condition B can be represented in the form

$$\eta_p(x) = \frac{1}{2}\sum_{i=1}^{p}\frac{\zeta_i^2(x)}{\mu_i^2} - \frac{p}{2},$$

where

$$\mu_1 = [1+(p-1)\rho_0]^{-1/2}, \quad \mu_2 = \cdots = \mu_p = [1-\rho_0]^{-1/2},$$

and $\zeta(x) = (\zeta_1(x), \ldots, \zeta_p(x))', x \in \mathbb{R}^n$, is a Gaussian vector field with independent components, $E\zeta(x) = 0$ and covariance matrix

$$E\zeta(0)\zeta(x)' = (\tilde{B}_{ij}(\|x\|))_{1\le i,j\le p},$$

$$\tilde{B}_{11}(\|x\|) = \mu_1^2[a(\|x\|) + (p-1)b(\|x\|)],$$

$$\tilde{B}_{ii}(\|x\|) = \mu_i^2[a(\|x\|) - b(\|x\|)], i = 2, \ldots, p,$$

and $\tilde{B}_{ij}(\|x\|) = 0, i \neq j$.

C. The spectral densities $f_i(\|\lambda\|)$, $i = 1, \ldots, p$ of the random fields $\zeta_i(x)$, $i = 1, \ldots, p$, exist and are decreasing for $\|\lambda\| \geq \lambda_0 \geq 0$ and continuous at all $\lambda \neq 0$.

Let us introduce the quantities

$$\theta_0 = \frac{1}{(2\pi)^{p/2}}\int_{\mathbb{R}^p}\exp\left\{-\frac{1}{4\mu}\sum_{i=1}^{p}\frac{u_i^2}{\mu_i^2}\right\}\exp\left\{-\frac{1}{2}\sum_{i=1}^{p}u_i^2\right\}du_1\ldots du_p$$
$$= \prod_{i=1}^{p}\left\{1+\frac{1}{2\mu\mu_i^2}\right\}^{-1/2}, \tag{2.18}$$

$$\theta_1 = \frac{1}{(2\pi)^{p/2}}\int_{\mathbb{R}^p}\left(u_1^2-1\right)\exp\left\{-\frac{1}{4\mu}\sum_{i=1}^{p}\frac{u_i^2}{\mu_i^2}\right\}\exp\left\{-\frac{1}{2}\sum_{i=1}^{p}u_i^2\right\}du_1\ldots du_p,$$

$$\theta_j = \frac{1}{(2\pi)^{p/2}}\int_{\mathbb{R}^p}\left(u_2^2-1\right)\exp\left\{-\frac{1}{4\mu}\sum_{i=1}^{p}\frac{u_i^2}{\mu_i^2}\right\}$$
$$\times\exp\left\{-\frac{1}{2}\sum_{i=1}^{p}u_i^2\right\}du_1\ldots du_p,\ j = 2, \ldots, p.$$

The following theorem was established in.[43]

Theorem 3. *Let $u(t, x)$, $(t, x) \in (0, \infty) \times \mathbb{R}^n$, be a solution of the initial value problem (2.1)–(2.2) with random initial condition $\eta(x) = \eta_p(x)$, $x \in \mathbb{R}^n$, satisfying Conditions B and C with $\alpha \in (0, n/2)$. Then the finite-dimensional distributions of the random fields*

$$Z_\varepsilon(t, x) = \frac{\varepsilon^{-(1+\alpha)/2}}{L(1/\sqrt{\varepsilon})} u\left(t/\varepsilon, x/\sqrt{\varepsilon}\right), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, 0 < \alpha < n/2,$$

converge weakly, as $\varepsilon \longrightarrow 0$, to the finite-dimensional distributions of the vector homogeneous (in x) random field

$$Z_2(t, x) = \sum_{j=1}^{p} \frac{\theta_j Y_j(t, x)}{2\theta_0}, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, 0 < \alpha < n/2, \tag{2.19}$$

where $Y_j(t, x)$, $j = 1, \ldots, p$, $(t, x) \in (0, \infty) \times \mathbb{R}^n$, are independent copies of the non-Gaussian random field $Z_1(t, x)$ defined by the representation (2.11), and θ_0, $\theta_1, \ldots, \theta_p$ are defined by the formulae (2.18).

We then have the following consequence of Theorem 2.

Theorem 4. *The random field $Z_2(t, x) = (Z_2^{(1)}(t, x), \ldots, Z_2^{(n)}(t, x))$, $(t, x) \in (0, \infty) \times \mathbb{R}^n$, defined by the representation (2.19) with fixed $t > 0$ is strictly stationary in x. Its moments of all orders exist and the corresponding cumulant spectra can be represented as*

$$f^{Z_2}_{l_1 \ldots l_k}(\lambda_1, \ldots, \lambda_{k-1}) = \sum_{j=1}^{p} \left(\frac{\theta_j}{2\theta_0}\right)^k f_{l_1 \ldots l_k}(\lambda_1, \ldots, \lambda_{k-1}),$$

where $f_{l_1 \ldots l_k}(\lambda_1, \ldots, \lambda_{k-1})$, $1 \le l_i \le n$, $i = 1, \ldots, k$, $k = 2, 3, \ldots$, are given by the formulae (2.13) and (2.14).

3. KPZ TURBULENCE PROBLEM

The KPZ equation describes an evolution of the profile of a growing interface (see[10,25,31,60] and the references therein). To introduce the KPZ turbulence problem, we first consider the following initial-value problem for the heat equation with external potential ϕ:

$$\frac{\partial h}{\partial t} = \mu \Delta h - h \cdot \phi \tag{3.1}$$

subject to the initial condition

$$h(0, x) = h_0(x) = \exp\left\{-\frac{\eta(x)}{2\mu}\right\}, \; x \in \mathbb{R}^n, \tag{3.2}$$

where $h = h(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, $\phi = \phi(x)$ and $\mu > 0$.

Introducing the transformation

$$\psi(t, x) = 2\mu \log h(t, x), t > 0, x \in \mathbb{R}^n,$$

we arrive at the following so-called KPZ equation

$$\frac{\partial \psi}{\partial t} = \mu \Delta \psi + \frac{1}{2} \|\nabla \psi\|^2 - 2\mu\phi \tag{3.3}$$

subject to the initial condition

$$\psi(0, x) = \psi_0(x) = -\eta(x),\ x \in \mathbb{R}^n. \tag{3.4}$$

Thus, for external potential $\phi \equiv 0$, we obtain the following solution to the intial-value problem (3.3)–(3.4):

$$\psi(t, x) = 2\mu \log \left[\int_{\mathbb{R}^n} \exp\left\{ -\frac{\|x - y\|^2}{4\mu t} \right\} \frac{1}{(4\pi\mu t)^{n/2}} e^{-\frac{\eta(y)}{2\mu}} dy \right], \tag{3.5}$$

which is naturally called the solution of the KPZ turbulence problem (3.3)–(3.4), if $\eta(x)$ is a measurable random field such that the integral (3.5) exists in the mean-square sense.

For $n = 1$, the KPZ turbulence problem (3.3)–(3.4) takes the form

$$\frac{\partial \psi}{\partial t} = \mu \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} \left(\frac{\partial \psi}{\partial x} \right)^2 - 2\mu\phi, \tag{3.6}$$

$$\psi(0, x) = -\eta(x), \tag{3.7}$$

and, by using some results of,[11] its solution can be written down for the cases of a linear external potential

$$\phi(x) = a + bx, \tag{3.8}$$

and a quadratic external potential

$$\phi(x) = a + bx^2, b > 0. \tag{3.9}$$

Indeed, for the linear external potential (3.8), the solution of the KPZ turbulence problem (3.6)–(3.7) is of the form

$$\psi(t, x) = -2\mu t(a + bx) + \frac{2\mu^2}{3} b^2 t^3 + 2\mu \log$$
$$\times \left[\int_{\mathbb{R}} \exp\left\{ -\frac{(x - y - b\mu t^2)^2}{4\mu t} \right\} \frac{1}{\sqrt{4\pi\mu t}} e^{\frac{-\eta(y)}{2\mu}} dy \right], \tag{3.10}$$

while for the quadratic external potential (3.9), the corresponding solution is of the form

$$\psi(t,x) = -2\mu a t + \sqrt{\mu} x^2 \tanh(\omega t) + 2\mu \log \left[\int_{\mathbb{R}} \exp\left\{ -\frac{[x - y\cosh(\omega t)]^2}{\sqrt{\mu/b}\sinh(2\omega t)} \right\} \right.$$
$$\left. \times \frac{1}{[2\pi\sqrt{\mu/b}\sinh(\omega t)]^{1/2}} e^{\frac{-\eta(y)}{2\mu}} dy \right], \tag{3.11}$$

if both stochastic integrals (3.10) and (3.11) exist in the mean-square sense, and

$$\omega = 2\sqrt{\mu b}. \tag{3.12}$$

Note that Batchelor *et al.*[12] have introduced for $n = 1$ a slightly different equation:

$$\frac{\partial \psi}{\partial t} = \mu \frac{\partial^2 \psi}{\partial x^2} + \nu \left(1 + \frac{1}{2} \left(\frac{\partial}{\partial x} \psi \right)^2 \right) + \mu \tag{3.13}$$

subject to initial condition

$$\psi(0,x) = -\eta(x), x \in \mathbb{R}, \tag{3.14}$$

where $\psi = \psi(t,x), t > 0, x \in \mathbb{R}, \mu > 0, \nu \in \mathbb{R}$.

The equation (3.13) differs from the standard form of the KPZ equation (3.6) with the external potential $\phi \equiv 0$. The constant velocity term in the equation (3.13) consists of two components; one arising from lateral growth ($\nu \neq 0$) and the other from vertical growth ($\mu \neq 0$). The equation (3.13) describes profile height in the evolution of smooth stromatolite laminae (with the surface roughness exponent equal to zero).

The general solution to the initial-value problem (3.13)–(3.14) can be obtained by first using the transform

$$\psi(t,x) = \frac{2\mu}{\nu} \log h(t,x)$$

and then using separation of variables.

The general solution to the KPZ type turbulence problem (3.13)–(3.14) is of the form

$$\psi(t,x) = \frac{2\mu}{\nu} \log \left[\int_{\mathbb{R}} \exp\left\{ -\frac{(x-y)^2}{4\mu t} \right\} \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{\nu}{2\mu}\eta(y)} dy \right] + (\mu + \nu)t, \tag{3.15}$$

if $\eta(x), x \in \mathbb{R}$ is a measurable stochastic process and the stochastic integral in (3.5) exists in the mean-square sense.

The scaling laws for both random fields (3.5) or (3.15) can be obtained from the corresponding scaling laws for the heat equation,[2–4,37] and Theorem 9 of Appendix C.

We now introduce the following condition concerning the initial velocity potential.

A$'$. Let $\eta(x), x \in \mathbb{R}^n$ be a measurable homogeneous and isotropic Gaussian random field with $E\eta(x) = 0$ and covariance function $B_\eta(x) = cov(\eta(0), \eta(x)), x \in \mathbb{R}^n$, such that

$$\int_{\mathbb{R}^n} \left|B_\eta(x)\right| dx < \infty, \int_{\mathbb{R}^n} B_\eta(x) dx \neq 0. \tag{3.16}$$

Theorem 5. *Let $\psi(t, x), t > 0, x \in \mathbb{R}^n$ be a KPZ-random field of the form (3.5), in which an initial potential $\eta(x), x \in \mathbb{R}^n$ is a random field satisfying one of the following conditions:*

*(a) $\eta(x)$ satisfies condition **A**$'$. (b) $\eta(x)$ satisfies condition **A**$'$ but instead of (3.16) its covariance function is of the form*

$$B_\eta(x) = \frac{L(\|x\|)}{\|x\|^\alpha}, 0 < \alpha < n,$$

as $\|x\| \to \infty$, where the function L is described in (2.8).

*(c) $\eta(x)$ satisfies condition **A**.*

*(d) $\eta(x)$ satisfies conditions **B** and **C**. Then, as $\varepsilon \to 0$, we have the following convergence of random fields in the sense of finite-dimensional distributions ($\overset{d}{\to}$):*

(i) when (a) holds

$$\frac{e^{\frac{1}{8\mu^2}}}{2\mu\varepsilon^{n/4}} \left[\psi\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}\right) - \frac{1}{4\mu}\right] \overset{d}{\to} X_1(t, x),$$

where $X_1(t, x), t > 0, x \in \mathbb{R}^n$ is the Gaussian random field with zero mean and covariance function

$$EX_1(t, x)X_1(t', x') = \sigma^2 \exp\left\{-\frac{\|x - x'\|^2}{4\mu(t + t')}\right\} \frac{1}{[4\pi\mu(t + t')]^{n/2}}, \tag{3.17}$$

$$\sigma^2 = \int_{\mathbb{R}^n} \left[\sum_{k=1}^{\infty} \frac{C_k^2}{k!} B_\eta^k(x)\right] dx,$$

$$C_k = \int_{\mathbb{R}^1} e^{-\frac{u}{2\mu}} \varphi(u) H_k(u) du, \quad \varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}},$$

$$H_k(u) = (-1)^k [\varphi(u)]^{-1} \frac{d^k}{du^k} \varphi(u) \quad being\ Hermite\ polynomials;$$

(ii) when (b) holds

$$\frac{e^{\frac{1}{8\mu^2}}}{2\mu\varepsilon^{\alpha/4}L^{1/2}\left(\frac{1}{\sqrt{\varepsilon}}\right)}\left[\psi\left(\frac{t}{\varepsilon},\frac{x}{\sqrt{\varepsilon}}\right)-\frac{1}{4\mu}\right]\overset{d}{\to}\ X_2(t,x),$$

where $X_2(t,x), t>0, x\in\mathbb{R}^n$ *is Gaussian random field with the following stochastic integral representation*

$$X_2(t,x)=-\frac{1}{2\mu}e^{\frac{1}{8\mu^2}}[c(n,\alpha)]^{1/2}\int_{\mathbb{R}^n}\frac{e^{i<x,\lambda>-\mu t\|\lambda\|^2}}{\|\lambda\|^{\frac{n-\alpha}{2}}}W(d\lambda),\ 0<\alpha<n, \quad (3.18)$$

and the covariance function

$$EX_2(t,x)X_2(t',x')=\frac{1}{4\mu^2}e^{\frac{1}{4\mu^2}}c(n,\alpha)\int_{\mathbb{R}^n}\frac{e^{i<x-x',\lambda>-\mu(t+t')\|\lambda\|^2}}{\|\lambda\|^{n-\alpha}}d\lambda, \quad (3.19)$$

with $c(n,\alpha)$ *being given by (2.10) and* $W(\cdot)$ *the Gaussian complex white noise random measure;*

(iii) when (c) holds

$$\frac{e^{\frac{1}{2\mu}\left(\frac{\mu}{1+\mu}\right)^{1/2}}}{2\mu\varepsilon^{\alpha/2}L\left(\frac{1}{\sqrt{\varepsilon}}\right)}\left[\psi\left(\frac{t}{\varepsilon},\frac{x}{\sqrt{\varepsilon}}\right)-1-\mu\log\frac{\mu}{1+\mu}\right]\overset{d}{\to}\ X_3(t,x),\ 0<\alpha<\frac{n}{2},$$

where $X_3(t,x), t>0, x\in\mathbb{R}^n$ *is the non-Gaussian random field with the following stochastic integral representation*

$$X_3(t,x)=\frac{c(n,\alpha)}{2}C(\mu)\int'_{\mathbb{R}^{2n}}\frac{e^{i<x,\lambda_1+\lambda_2>-\mu t\|\lambda_1+\lambda_2\|^2}}{(\|\lambda_1\|\cdot\|\lambda_2\|)^{\frac{n-\alpha}{2}}}W(d\lambda_1)W(d\lambda_2), \quad (3.20)$$

the double stochastic integral $\int'_{(\cdot)}$ *is evaluated with respect to complex white noise Gaussian random measure with the diagonal hyperplanes* $\lambda_1=\pm\lambda_2$ *being excluded from the domain of integration and*

$$C(\mu)=e^{\frac{1}{2\mu}}\left[\left(\frac{\mu}{1+\mu}\right)^{3/2}-\left(\frac{\mu}{1+\mu}\right)^{1/2}\right]=-e^{\frac{1}{2\mu}}\frac{\mu^{1/2}}{(1+\mu)^{3/2}}.$$

(iv) when (d) holds

$$\frac{\theta_0}{2\mu\varepsilon^{\alpha/2}L\left(\frac{1}{\sqrt{\varepsilon}}\right)}\left[\psi\left(\frac{t}{\varepsilon},\frac{x}{\sqrt{\varepsilon}}\right)-2\mu\log\theta_0\right]\overset{d}{\to}\ X_4(t,x)$$

$$=\sum_{j=1}^{p}\frac{\theta_j X_3^{(j)}(t,x)}{2\theta_0},\ 0<\alpha<\frac{n}{2}, \quad (3.21)$$

where $X_3^{(j)}(t,x)$, $j = 1, \ldots, p$, $t > 0$, $x \in \mathbb{R}^n$, *are independent copies of the non-Gaussian field (3.20) and* θ_j, $j \geq 0$ *are defined in (2.18).*

Remark 3. The scaling laws for the random field (3.10) can be obtained from Theorem 5 with $n = 1$ and the identity

$$\widetilde{h}\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}} + \frac{\mu b t^2}{\varepsilon^2}\right) = R_\varepsilon(t,x)\int_{\mathbb{R}} h\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}} - y\right) h_0(y)dy,$$

where

$$h(t,x) = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{x^2}{4\mu t}}$$

and

$$\widetilde{h}(t,x) = \exp\left\{-(a+bx)t + \frac{\mu b^2 t^3}{3}\right\}\int_{\mathbb{R}} e^{-\frac{(x-y-b\mu t^2)^2}{4\mu t}} \frac{h_0(y)}{\sqrt{4\pi\mu t}} dy$$

is the solution of the initial-value problem (3.1)–(3.2) with the linear potential (3.8), and

$$R_\varepsilon(t,x) = \exp\left\{-\frac{ta}{\varepsilon} - \frac{bxt}{\varepsilon\sqrt{\varepsilon}} + \frac{\mu b^2 t^3}{3}\right\}.$$

For instance, under the conditions of Theorem *5,*

$$\frac{e^{\frac{1}{8\mu^2}}}{2\mu R_\varepsilon(t,x) A_i(\varepsilon)}\left[\psi\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}} + \frac{\mu b t^2}{\varepsilon^2}\right) - \frac{1}{4\mu} + \frac{ta}{\varepsilon} + \frac{bxt}{\varepsilon\sqrt{\varepsilon}} - \frac{\mu b^2 t^3}{3}\right]$$

$$\overset{d}{\to} X_i(t,x),\ t > 0,\ x \in \mathbb{R}^1,\ i = 1,2,3,4,$$

where

$$A_1(\varepsilon) = \varepsilon^{1/4},\ A_2(\varepsilon) = \varepsilon^{\alpha/4} L^{1/2}\left(\frac{1}{\sqrt{\varepsilon}}\right), 0 < \alpha < 1,$$

$$A_3(\varepsilon) = A_4(\varepsilon) = \varepsilon^{\alpha/2} L\left(\frac{1}{\sqrt{\varepsilon}}\right), 0 < \alpha < 1/2. \tag{3.22}$$

Remark 4. The scaling laws for the random fields (3.11) are more difficult and will be discussed elsewhere.

Remark 5. The scaling law for the random field (3.15) can be obtained in a similar manner to Theorem 5 using Theorem 9 (see Appendix C) with $g(x) = \frac{2\mu}{\nu}\log x + (\mu+\nu)t$, $g'(x) = \frac{2\mu}{\nu x}$. In fact, by putting $n = 1$ and replacing the term

$[\psi(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}) - \frac{1}{4\mu}]$ by $[\psi(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}) - (\mu+\nu)\frac{t}{\varepsilon} - \frac{1}{4\mu\nu}]$ and the factor 2μ in the right hand side by $\frac{2\mu}{\nu}$, we obtain under the conditions of Theorem 5 that

$$\frac{\nu e^{-\frac{1}{8\mu^2}}}{2\mu A_i(\varepsilon)}\left[\psi\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}\right) - (\mu+\nu)\frac{t}{\varepsilon} + \frac{1}{4\mu\nu}\right] \overset{d}{\to} X_i(t,x),\, i = 1, 2, 3, 4,$$

where $A_i(\varepsilon)$ are described in *(3.22)*.

The second-order and higher-order spectral densities can now be obtained from Theorem 5 and the results of.[6] Indeed, the second-order spectral density of the Gaussian field $X_1(t,x)$ is of the form

$$f_{1,2}(\lambda) = \frac{\sigma^2}{(2\pi)^n}\exp\{-\|\lambda\|^2 2\mu t\},\, \lambda \in \mathbb{R}^n,$$

where σ^2 is defined in (3.17), while the second-order spectral density of the Gaussian field $X_2(t,x)$ is of the form

$$f_{2,2}(\lambda) = \frac{1}{4\mu^2} e^{\frac{1}{4\mu^2}} c(n,\alpha)\frac{e^{-2\mu t\|\lambda\|^2}}{\|\lambda\|^{n-\alpha}},\, 0 < \alpha < n,\ \lambda \in \mathbb{R}^n.$$

The second-order spectral density of the non-Gaussian random field $X_3(t,x)$ is of the form

$$f_{3,2}(\lambda) = C^2(\mu)\frac{c^2(n,\alpha)}{2}k(\alpha)\frac{e^{-2\mu t\|\lambda\|^2}}{\|\lambda\|^{n-2\alpha}},\quad 0 < \alpha < \frac{n}{2},\ \lambda \in \mathbb{R}^n, \tag{3.23}$$

$k(\alpha)$ being given by (2.15), while the third-order spectral densities of the non-Gaussian random field $X_3(t,x)$ is of the form

$$f_{3,3}(\lambda_1,\lambda_2) = c^3(n,\alpha)C(\mu)^3 \tag{3.24}$$

$$\times \operatorname*{sym}_{\lambda_1,\lambda_2,\lambda_3:\lambda_1+\lambda_2+\lambda_3=0}\left[\exp\left\{-\mu t(\|\lambda_1\|^2 + \|\lambda_2\|^2 + \|\lambda_1+\lambda_2\|^2)\right\} g_3(\lambda_1,\lambda_2)\right],$$

where

$$g_3(\lambda_1,\lambda_2) = \int_{\mathbb{R}^n}\frac{dz}{(\|\lambda_1+\lambda_2+z\|\,\|\lambda_2+z\|\,\|z\|)^{n-\alpha}},\quad 0 < \alpha < \frac{n}{2},\quad \lambda_1,\lambda_2 \in \mathbb{R}^n.$$

The function $g_3(\lambda_1,\lambda_2)$ is homogeneous of order $H = 3\alpha - 2n$, that is, $g_3(t\lambda_1, t\lambda_2) = t^H g_3(\lambda_1,\lambda_2)$, and its Fourier transform is given by

$$\widehat{g_3}(\zeta_1,\zeta_2) = \left(\pi^{\frac{n}{2}-\alpha}\frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}\right)^3 (\|\zeta_1\|\,\|\zeta_2\|\,\|\zeta_1-\zeta_2\|)^{-\alpha}. \tag{3.25}$$

The asymptotic behavior of the functions $g_3(\lambda_1,\lambda_2)$ as $\lambda_i \to 0$ is described in Remark 1 in Sec. 2.

The corresponding trispectra are more complicated. We are able to obtain

Theorem 6. *The random field $X_3(t,x)$, $t>0$, $x\in\mathbb{R}^n$, with fixed $t>0$ is strictly stationary in x of the fourth order with $E\,|X_3(t,x)|^4<\infty$ and its trispectra $f_{3,4}(\lambda_1,\lambda_2,\lambda_3)$ can be expressed as*

$$f_{3,4}(\lambda_1,\lambda_2,\lambda_3)=3c^4(n,\alpha)\,C^4(\mu)$$

$$\times \operatorname*{sym}_{\{\lambda_1,\lambda_2,\lambda_3,\lambda_4:\Sigma_{i=1}^4\lambda_i=0\}}\left[\exp\left\{-\mu t\left(\sum_{i=1}^{3}\|\lambda_i\|^2+\left\|\sum_{i=1}^{3}\lambda_i\right\|^2\right)\right\}g_4(\lambda_1,\lambda_2,\lambda_3)\right],$$

where

$$g_4(\lambda_1,\lambda_2,\lambda_3)=\int_{\mathbb{R}^n}\|\lambda_1+\lambda_2+\lambda_3$$
$$+\mu\|^{\alpha-n}\,\|\lambda_1+\lambda_2+\mu\|^{\alpha-n}\,\|\lambda_1+\mu\|^{\alpha-n}\,\|\mu\|^{\alpha-n}\,d\mu. \qquad (3.26)$$

In general, the spectral densities of an arbitrary order p for the field $X_3(t,x)$ are presented in the next theorem.

Theorem 7. *The random field $X_3(t,x)$, $t>0$, $x\in\mathbb{R}^n$, defined by the representation (3.20) for fixed $t>0$, is strictly stationary in x of order p with $E\,|X_3(t,x)|^p<\infty$. Its spectral densities of order p, $f_{3,p}\left(\lambda_1,\ldots,\lambda_{p-1}\right)$, can be represented in the form*

$$f_{3,p}\left(\lambda_1,\ldots,\lambda_{p-1}\right)=c^p(n,\alpha)\,C^p(\mu)2^{-1}(p-1)! \qquad (3.27)$$

$$\times \operatorname*{sym}_{\{\lambda_1,\ldots,\lambda_p:\Sigma_{i=1}^p\lambda_i=0\}}\left[e^{-\mu t\left(\Sigma_{i=1}^{p-1}\|\lambda_i\|^2+\left\|\Sigma_{i=1}^{p-1}\lambda_i\right\|^2\right)}g_p(\lambda_1,\ldots,\lambda_{p-1})\right],$$

where

$$g_p(\lambda_1,\ldots,\lambda_{p-1})=\int_{\mathbb{R}^n}\frac{d\lambda}{\left(\|\lambda\|\,\|\lambda+\lambda_1\|\ldots\left\|\lambda+\Sigma_{i=1}^{p-1}\lambda_i\right\|\right)^{n-\alpha}}. \qquad (3.28)$$

It remains to describe the spectral densities of the field $X_4(t,x)$, which appears in the case (iv) of Theorem 5. This is done in the next theorem.

Theorem 8. *The random field $X_4(t,x)$, $t>0$, $x\in\mathbb{R}^n$, defined by the representation in the right hand side of the formula (3.21) for fixed $t>0$ is strictly stationary in x. Its moments of all orders exist and the corresponding spectral* *densities of order $k \geq 2$ can be represented as*

$$f_{4,k}(\lambda_1, \ldots, \lambda_{k-1}) = \sum_{j=1}^{p} \left(\frac{\theta_j}{2\theta_0} \right)^k f_{3,k}(\lambda_1, \ldots, \lambda_{k-1}),$$

where $f_{3,k}(\lambda_1, \ldots, \lambda_{k-1})$ are given by the formulae (3.27).

4. PROOFS

Proof. [Proof of Theorem 2]:

(a) We have

$$Cov\big(Z_1^{(l)}(t,x), Z_1^{(k)}(t,x+y)\big) = \frac{\mu^4}{(1+\mu)^2}(c(n,\alpha))^2$$

$$\times \int_{\mathbb{R}^{2n}} \frac{e^{i(y,\lambda_1+\lambda_2)-2\mu t\|\lambda_1+\lambda_2\|^2}\big(\lambda_1^{(l)}+\lambda_2^{(l)}\big)\big(\lambda_1^{(k)}+\lambda_2^{(k)}\big)}{(\|\lambda_1\|\,\|\lambda_2\|)^{n-\alpha}} d\lambda_1 d\lambda_2.$$

The change of variables $\lambda_1 = \lambda_1' - \lambda_2',\ \lambda_2 = \lambda_2'$ yields $\lambda_1 + \lambda_2 = \lambda_1'$ and

$$\int_{\mathbb{R}^{2n}} \frac{e^{i(y,\lambda_1+\lambda_2)-2\mu t\|\lambda_1+\lambda_2\|^2}(\lambda_1^{(l)}+\lambda_2^{(l)})(\lambda_1^{(k)}+\lambda_2^{(k)})}{(\|\lambda_1\|\,\|\lambda_2\|)^{n-\alpha}} d\lambda_1 d\lambda_2$$

$$= \int_{\mathbb{R}^{2n}} e^{i(y,\lambda_1)} \frac{e^{-2\mu t\|\lambda_1\|^2}\lambda_1^{(l)}\lambda_1^{(k)}}{(\|\lambda_1-\lambda_2\|\,\|\lambda_2\|)^{n-\alpha}} d\lambda_1 d\lambda_2$$

$$= \int_{\mathbb{R}^n} e^{i(y,\lambda)} \left(e^{-2\mu t\|\lambda\|^2}\lambda^{(l)}\lambda^{(k)} \int_{\mathbb{R}^n} \frac{d\lambda_2}{(\|\lambda-\lambda_2\|\,\|\lambda_2\|)^{n-\alpha}} \right) d\lambda.$$

From the Riesz composition formula (see Appendix A) we have

$$\int_{\mathbb{R}^n} \frac{d\lambda_2}{(\|\lambda-\lambda_2\|\,\|\lambda_2\|)^{n-\alpha}} = k(\alpha)\,\|\lambda\|^{2\alpha-n}.$$

From the above formulae it follows that the spectral densities of second order are given by

$$f_{lk}(\lambda) = \frac{\mu^4 c^2(n,\alpha)}{(1+\mu)^2} k(\alpha) e^{-2\mu t\|\lambda\|^2}\|\lambda\|^{2\alpha-n}\lambda^{(l)}\lambda^{(k)}.$$

(b) Using the diagram formula (see Appendix B), the p-th cumulant can be written as

$$cum\left(Z_1^{(l_1)}(t,x_1), Z_1^{(l_2)}(t,x_2), \ldots, Z_1^{(l_p)}(t,x_p)\right) = \left(\frac{\mu^2 i}{1+\mu}c(n,\alpha)\right)^p \sum_{\gamma\in\Gamma_p^c} h_\gamma, \tag{4.1}$$

where Γ_p^c is the set of all complete closed diagrams γ with p levels over vertices $(n_1, \ldots, n_p) = (2, \ldots, 2)$, all diagrams $\gamma \in \Gamma_p^c$ are "circular", that is, the vertices $(1,1), (1,2), (2,1), (2,2), \ldots, (p,1), (p,2)$ can be ordered as $(1,1)$, $(1,2), (\pi_2, i_3), (\pi_2, i_4), \ldots, (\pi_p, i_{2p-1}), (\pi_p, i_{2p})$ with $(1,2)$ connected to (π_2, i_3) by an edge in γ, (π_k, i_{2k}) connected to (π_{k+1}, i_{2k+1}), $k = 2, \ldots, p-1$ and (π_p, i_{2p}) connected to $(1,1)$, and

$$h_\gamma = \int_{\mathbb{R}^{np}} h_{x_1}^{(l_1)}(\lambda_1', \lambda_2) h_{x_{\pi_2}}^{(l_{\pi_2})}(\lambda_2', \lambda_3) \ldots h_{x_{\pi_p}}^{(l_{\pi_p})}(\lambda_p', \lambda_1) \times \prod\nolimits_{i=1}^{p} \delta(\lambda_i' + \lambda_i) d\lambda_1 \ldots d\lambda_p,$$

$$h_x^{(l)}(\lambda_1, \lambda_2) = \frac{e^{i(x, \lambda_1 + \lambda_2) - \mu t \|\lambda_1 + \lambda_2\|^2} \left(\lambda_1^{(l)} + \lambda_2^{(l)}\right)}{(\|\lambda_1\| \, \|\lambda_2\|)^{(n-\alpha)/2}},$$

$(\pi_2, \ldots, \pi_p)$ being a permutation of $(2, \ldots, p)$. The number of all circular diagrams is $\left|\Gamma_p^c\right| = 2^{p-1}(p-1)!$. Hence there are $2^{p-1}(p-1)$ terms in the sum on the right-hand side of (4.1). The typical term to estimate is

$$\int_{\mathbb{R}^{np}} h_{x_1}^{(l_1)}(-\lambda_1, \lambda_2) \, h_{x_2}^{(l_2)}(-\lambda_2, \lambda_3) \ldots h_{x_p}^{(l_p)} \times (-\lambda_p, \lambda_1) d\lambda_1 \ldots d\lambda_p = c_p(x_1, \ldots x_p).$$

We have

$$c_p\left(x_1, \ldots x_p\right) = \int_{\mathbb{R}^{np}} e^{i(x_1, \lambda_2 - \lambda_1) + (x_2, \lambda_3 - \lambda_2) + \ldots + \left(x_p, \lambda_1 - \lambda_p\right)} \times \frac{e^{-\mu t (\|\lambda_2 - \lambda_1\|^2 + \|\lambda_3 - \lambda_2\|^2 + \ldots + \left\|\lambda_1 - \lambda_p\right\|^2)}}{(\|\lambda_1\| \ldots \|\lambda_p\|)^{n-\alpha}} \times \left(\lambda_2^{(l_1)} - \lambda_1^{(l_1)}\right)\left(\lambda_3^{(l_2)} - \lambda_2^{(l_2)}\right) \ldots \left(\lambda_1^{(l_p)} - \lambda_p^{(l_p)}\right) d\lambda_1 \ldots d\lambda_p.$$

Noting that $(x_1, \lambda_2 - \lambda_1) + (x_2, \lambda_3 - \lambda_2) + \ldots + (x_p, \lambda_1 - \lambda_p) = (x_1 - x_p, \lambda_2 - \lambda_1) + (x_2 - x_p, \lambda_3 - \lambda_2) + \ldots + (x_{p-1} - x_p, \lambda_p - \lambda_{p-1})$, the change of variables $\lambda_k - \lambda_{k-1} = \lambda_k'$, $k = 2, \ldots, p$ and $\lambda_1 = \lambda_1'$ yields

$$c_p\left(x_1, \ldots x_p\right) = \int_{\mathbb{R}^{(p-1)n}} e^{i\left(\left(x_1 - x_p, \lambda_2\right) + \left(x_2 - x_p, \lambda_3\right) + \ldots + \left(x_{p-1} - x_p, \lambda_p\right)\right)}$$

$$\times \left\{ e^{-\mu t \left(\|\lambda_2\|^2 + \|\lambda_3\|^2 + \ldots + \left\|\lambda_p\right\|^2 + \left\|\Sigma_{k=2}^p \lambda_k\right\|^2\right)} \lambda_2^{(l_1)} \lambda_3^{(l_2)} \ldots \lambda_p^{(l_{p-1})} \left(\Sigma_{k=2}^p \lambda_k^{(l_p)}\right) \times \int_{\mathbb{R}^n} \frac{d\lambda_1}{\left(\|\lambda_1\| \, \|\lambda_1 + \lambda_2\| \ldots \left\|\lambda_1 + \lambda_2 + \ldots + \lambda_p\right\|\right)^{n-\alpha}} \right\} d\lambda_2 \ldots d\lambda_p.$$

Hence, the p-th order spectral density of the process $Z_1(t, x)$ is given by the following formula:

$$f_{l_1 \ldots l_p}\left(\lambda_1, \ldots, \lambda_{p-1}\right) = \left(\frac{\mu^2 i}{(1+\mu)}\right)^p c^p(n,\alpha)\, 2^p(p-1)$$
$$\times \underset{\{\lambda_1,\ldots,\lambda_p : \Sigma_{i=1}^p \lambda_i = 0\}}{\mathrm{sym}} \left\{ \exp\left\{-\mu t \left(\sum_{i=1}^{p-1} \|\lambda_i\|^2 + \|\sum_{i=1}^{p-1} \lambda_i\|^2\right)\right\}\right.$$
$$\times \lambda_1^{(l_1)} \lambda_2^{(l_2)} \ldots \lambda_{p-1}^{(l_{p-1})} \left(\sum_{i=1}^{p-1} \lambda_i^{(l_p)}\right)$$
$$\left. \times \int_{\mathbb{R}^n} \frac{d\lambda}{\left(\|\lambda\| \, \|\lambda + \lambda_1\| \ldots \left\|\lambda + \Sigma_{i=1}^{p-1} \lambda_i\right\|\right)^{n-\alpha}} \right\}.$$

Proof. [Proof of Theorem 4]: Theorem 4 follows as a consequence of Theorem 2 and the following properties of cumulants:
(i) $cum\{a_1 X_1, \ldots, a_n X_n\} = a_1 \ldots . a_n \, cum\{X_1, \ldots, X_n\}$ for constants $a_1, \ldots, a_n$;
(ii) if the random vectors $(X_1, \ldots, X_n)'$ and $(Y_1, \ldots, Y_n)'$ are statistically independent, then

$$cum\{X_1 + Y_1, \ldots, X_n + Y_n\} = cum\{X_1, \ldots, X_n\} + cum\{Y_1, \ldots, Y_n\}.$$

The proofs of the theorems of Sec. 3 can be obtained analogously, with the use of the same technique as above (see also Anh *et al.* 2003), hence will be omitted.

APPENDIX A: RIESZ'S COMPOSITION FORMULA

The following statement is known as Riesz's composition formula:
Suppose that $0 < \alpha < n,\ 0 < \beta < n,\ 0 < \alpha + \beta < n$, then

$$\int_{\mathbb{R}^n} \|x - z\|^{\alpha - n} \|z - y\|^{\beta - n} dz = k(\alpha, \beta) \|x - y\|^{\alpha + \beta - n},$$

where

$$k(\alpha, \beta) = \pi^{n/2} \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\beta}{2}\right) \Gamma\left(\frac{n-\alpha-\beta}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right) \Gamma\left(\frac{n-\beta}{2}\right) \Gamma\left(\frac{\alpha+\beta}{2}\right)} \tag{A.1}$$

(see[21], p. 71).

APPENDIX B: CUMULANTS OF MULTIPLE STOCHASTIC INTEGRALS

This Appendix is based on.[20,22,57,58]

One of the basic tools for evaluating products of multiple stochastic Wiener-Itô integrals and their moments is the diagram formula. It originates from the diagram formula for the products of Hermite polynomials of Gaussian random variables. We prepare here the formula for evaluating the cumulants of multiple stochastic integrals which is a consequence of the diagram formula.

We first introduce some notations and definitions.

Let $m_1, \ldots, m_p$ be given positive integers. An undirected graph Γ with $m_1 + \ldots + m_p = M$ vertices is called a diagram of order $(m_1, \ldots, m_p)$ if

a) the set of vertices V of the graph Γ is of the form

$$V = \{(1, 1), \ldots, (1, m_1), (2, 1), \ldots, (2, m_2), \ldots, (p, 1), \ldots, (p, m_p)\}$$
$$= \bigcup_{j=1}^{p} W_j, \qquad \text{(B.1)}$$

where

$$W_j = \{(j, l) : 1 \leq l \leq m_j\}$$

is the j-th level of the graph Γ, $1 \leq j \leq p$;

b) each vertex is at most of degree 1, that is, met by at most one edge;

c) if vertices (j_1, i_1) and (j_2, i_2) are joined by an edge $w = ((j_1, i_1), (j_2, i_2))$, then $j_1 \neq j_2$, that is, the edges of the graph Γ can connect only different levels.

Let $\Gamma(m_1, \ldots, m_p)$ denote the set of diagrams of order $(m_1, \ldots, m_p)$. Denote by $\mathcal{K}(\gamma)$ the set of edges of a diagram $\gamma \in \Gamma(m_1, \ldots, m_p)$. With each element $v \in V$, we can associate an integer denoting the position at which v appears in the list (B.1). Thus the position of $(1, 1)$ is 1, the positions of $(1, 2)$ is 2 and so on. The position of the last vertex (p, m_p) is M. Each edge $w = ((j_1, i_1), (j_2, i_2)) \in \mathcal{K}(\gamma)$ can also be thought of as $w = (k_1, k_2)$, where k_1 is the position of the vertex (j_1, i_1) and k_2 is the position of the vertex (j_2, i_2) in the list (B.1). A diagram γ is called *complete* if each of its vertices is met by an edge, that is, there exist no isolated vertices. In such a case, the number of edges in γ is $|\mathcal{K}(\gamma)| = M/2$. A diagram is called *closed* if the set of its levels $\{W_j, j = 1, \ldots, p\}$ cannot be split into two subsets connected by no edge.

Let $h_i \in L_2(\mathbb{R}^{nm_i}), i = 1, \ldots, p$, and define

$$h(\lambda_1, \ldots, \lambda_M) = \prod_{i=1}^{p} h_i(\lambda_{M_{i-1}+1}, \ldots, \lambda_{M_i}),$$

where $M_i = m_1 + \ldots + m_i$, $i = 1, 2, \ldots, p$, $M_0 = 0$ and $M_p = M$. The following formula is used in the proof of Theorem 2:

$$cum\left(\int'_{\mathbb{R}^{nm_1}} h_1\left(\lambda_1, \ldots, \lambda_{m_1}\right) \prod_{i=1}^{m_1} W\left(d\lambda_i\right), \right.$$
$$\left. \ldots, \int'_{\mathbb{R}^{nm_p}} h_p\left(\lambda_1, \ldots, \lambda_{m_p}\right) \prod_{i=1}^{m_p} W\left(d\lambda_i\right)\right)$$

$$= \sum_{\gamma \in \Gamma^c\left(m_1, \ldots, m_p\right)} \int_{\mathbb{R}^{nM/2}} h\left(\lambda_1, \ldots, \lambda_M\right) \prod_{\left(k_i, k_j\right) \in \mathcal{K}(\gamma)} \left\{\delta\left(\lambda_{k_i} + \lambda_{k_j}\right) d\lambda_{k_i}\right\}, \qquad \text{(B.2)}$$

where the sum is taken over all complete closed diagrams γ of order $\left(m_1, \ldots, m_p\right)$, $\mathcal{K}(\gamma)$ is the set of edges of the diagrams γ, and $\delta(\cdot)$ is the Kronecker delta function.

APPENDIX C: DELTA METHOD

The following statement can be proved similarly to Serfling,[(54)] pp. 122-123 (see also,[(50)] pp. 262-263 for a new proof of this result by using the Skorokhod theorem for limiting normal law. The proof does not depend on the limiting random variable).

Theorem 9. *Let $h(t, x), t > 0, x \in \mathbb{R}^n$ be a spatiotemporal random field such that for some real function $m(t, x)$ and some function A_ε we have the following convergence of finite-dimensional distributions:*

$$\frac{1}{A_\varepsilon}\left[u\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}\right) - m\right] \overset{d}{\to} U(t, x)$$

as $\varepsilon \to 0$, where $U(t, x), t > 0, x \in \mathbb{R}^n$ is a spatiotemporal random field.

Then for any real-valued function $g(u), u \in \mathbb{R}^1$ differentiable at $u = m$, with $g'(m) \neq 0, m \in \mathbb{R}$, the following convergence of finite-dimensional distributions holds true:

$$\frac{1}{|g'(m)|\, A_\varepsilon}\left[g\left(u\left(\frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}}\right)\right) - g(m)\right] \overset{d}{\to} U(t, x)$$

as $\varepsilon \to 0$.

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